Indian Statistical Institute, Bangalore Centre. Mid-Semester Exam : Graph Theory

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Max. points : 40.

Time Limit : 2.5 hours.

There are two parts to the question paper - PART A and PART B. Read the instructions for each section carefully. Some common notations and definitions are listed on Page 3.

1 PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

Please write only the correct choice(s) (for ex., (a), (b) et al.) in your answer scripts. No explanations are needed. Write PART A answers in a separate page.

Some questions will have multiple correct choices. Answer all questions. Each question carries 2 points.

- 1. Consider the hypercube graph on $\{0,1\}^n$. Which of the following are true about the hypercube graph ?
 - (a) The diameter is n/2.
 - (b) It is a bi-partite graph.
 - (c) It has a perfect matching.
 - (d) It is connected.
- 2. Which of the following are true about complete graph on n vertices ?
 - (a) It is a bi-partite graph for all $n \ge 1$.
 - (b) It has a perfect matching for all $n \ge 1$.
 - (c) The diameter is 1.
 - (d) It has n^{n-3} spanning trees.
- 3. Which of the following are true about the cycle graph C_n on n vertices ?
 - (a) It is a bi-partite graph for all $n \ge 1$.
 - (b) It has chromatic number of at most 3.
 - (c) It has a perfect matching for all $n \ge 1$.

- (d) It has diameter at least (n-1)/2.
- 4. Which of the following are true about the path graph P_n on n vertices ?
 - (a) It is a bi-partite graph for all $n \ge 1$.
 - (b) It is a tree
 - (c) It has a perfect matching for all $n \ge 1$.
 - (d) It has diameter n.
- 5. Which of the following are true about the Petersen graph?
 - (a) It is a bi-partite graph.
 - (b) It has a perfect matching.
 - (c) It has 16 edges.
 - (d) It has diameter 2.

2 PART B : 30 Points.

Answer any three questions only. All questions carry 10 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. Always define the underlying random variables and events clearly before computing anything !

- 1. Compute the eigenvalues of the cycle graph C_n for all $n \ge 3$ and use this to find number of spanning trees of the wheel graph W_{n+1} , $n \ge 3$.
- 2. Compute the chromatic polynomial of the wheel graph W_{n+1} , $n \ge 3$ and the complete bi-partite graph $K_{n,m}$, $n \ge 1$, $m \ge 1$.
- 3. Let G be a graph with at least 3 vertices. Show that G is 2-connected (i.e., vertex connected) $\Leftrightarrow G$ is connected and has no cut-vertex \Leftrightarrow for all $x, y \in V(G)$ there exists a cycle through x and $y \Leftrightarrow \delta(G) \ge 1$ and every pair of edges lies in a common cycle.
- 4. Let G be an undirected graph with vertex set $\{s, t, 2, ..., 7\}$ with the following edge capacities : c(s, 2) = 10, c(s, 3) = 13, c(2, 5) = 7, c(3, 4) = 5, c(3, 7) = 6, c(4, 5) = 3, c(4, 6) = 3, c(5, 6) = 2, c(5, t) = 9, c(6, 7) = 3, c(6, t) = 4, c(7, t) = 10. We have used c(u, v) to denote capacity for edge (u, v). The edge-capacities which are not specified are assumed to be 0. As usual, s denotes the source and t denotes the target.

(a) Draw the network with the capacities on each edge marked.(2)

(b) Run the Ford-Fulkerson algorithm to find the maximum flow¹. Draw/mention the flows at each step of the algorithm. (5)

- (c) Describe a minimal cut. (3)
- 5. Show the following.

(a) Show that a graph with n vertices is a tree iff $P(G, x) = x(x-1)^{n-1}$. (5)

¹Here the flow is on the undirected graph G i.e., satisfies anti-symmetry, Kirchoff's node law and non-negative output at the source s

(b) Let $e = (v, w) \in E(G)$ be any edge. Show that $\tau(G) = \tau(G - e) + \tau(G/e)$ where $\tau(G)$ is the number of spanning trees of G. Here G - e denotes edge deletion and G/e denotes edge contraction. (5)

Some notations :

- G is assumed to be a finite simple graph everywhere.
- d_G is defined as the usual graph metric when all edge weights are taken to be 1 and $diam(G) := \max\{d_G(u, v) : u, v \in V(G)\}.$
- Q_n is the hypercube graph on $\{0,1\}^n$ i.e., $V = \{0,1\}^n$ and $x \sim y$ is x and y differ exactly in one-coordinate i.e., $|\{i: x_i \neq y_i\}| = 1$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.
- The wheel graph W_{n+1} is the graph obtained by adding a new vertex to C_n and connecting it to all the *n* vertices of C_n .
- The complete bi-partite graph $K_{n,m}$ is a bi-partite graph on $V = V_1 \sqcup V_2$ with $|V_1| = n, |V_2| = m$ and edges between any two vertices in V_1 and V_2 .
- A vertex $v \in G$ is a cut-vertex if $\beta_0(G v) > \beta_0(G)$ where β_0 is the number of connected components.
- P(G, x) is the chromatic polynomial of G.